RESEARCH NOTES

A generalized wave action conservative equation for the dissipative dynamical system in the nearshore region*

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Abstract To describe the various complex mechanisms of the dissipative dynamical system between waves, currents, and bottoms in the nearshore region that induce typically the wave motion on large-scale variation of ambient currents, a generalized wave action equation for the dissipative dynamical system in the nearshore region is developed by using the mean-flow equations based on the Navier-Stokes equations of viscous fluid, thus raising two new concepts: the vertical velocity wave action and the dissipative wave action, extending the classical concept, wave action, from the ideal averaged flow conservative system into the real averaged flow dissipative system (that is, the generalized conservative system). It will have more applications.

Keywords: dissipative dynamical system, wave action conservation, the averaged flow equations, the Navier-Stokes equations.

The interaction of waves and currents and bottoms is ubiquitous in coastal seas[1], evolving into a typical dissipative dynamical system in nature with multiple dissipative factors, such as wave breaking, bottom friction and the free surface stress induced by wind. The system itself is often treated as non-dissipative, thus deriving a number of conservation laws[2], among which the wave action law[3~7] is particularly of more generality and insight than the other ones, and the wave action becomes one of essential concepts in hydrodynamics.

Until now almost all classical processes observed in the physical world are irreversible dissipative dynamical systems[8], therefore it is essential by now that some classical conservative laws should be reconsidered and investigated to make a certain extension for real dissipative dynamical systems[9~11]. The aim of the present work is to develop a generalized wave action conservative equation, and further elucidate the physical mechanisms of wave-current-bottom interactions, by proceeding from the most general framework of the viscous Navier-Stokes equations of motion.

1 The averaged equations of motion

There are two main approaches to describing waves on a current with large-scale variation. One is the use of an averaged Lagrangian developed by Whitham[12], the other is direct integration with respect to vertical coordinate and averaging of equations of motion[13~15], which is easier to appreciate the physical significance of terms and hence to make appropriate additions to the equations in order to account for wave dissipation, wave generation, or even wave breaking, and given as follows[15].

Consider flow of an incompressible viscous fluid as being at the center of fluid dynamics by virtue of its fundamental nature and its practical importance. Using $\boldsymbol{x}\equiv(x, y)\equiv(x_1, x_2)$ as the horizontal coordinates and z the vertical coordinates, we denote the sea bed at $z=-h(\boldsymbol{x})$ and the free surface displacement as $z=\zeta(\boldsymbol{x})$. Dividing ζ and the velocity, $\boldsymbol{V}=(u_1, u_2, w)$, into mean and fluctuating parts yields

$$\zeta(\boldsymbol{x}, t)=\langle\zeta(\boldsymbol{x}, t)\rangle+\bar{\zeta}(\boldsymbol{x}, t), \quad \langle\zeta(\boldsymbol{x}, t)\rangle=0, \tag{1}$$

$$u_i(\boldsymbol{x}, z, t)=U_i(\boldsymbol{x}, t)+\tilde{u}_i(\boldsymbol{x}, z, t),$$

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$$w(\boldsymbol{x}, z, t) = W(\boldsymbol{x}, t) + \tilde{w}(\boldsymbol{x}, z, t), \quad (2)$$

where $\langle \cdot \rangle$ denotes time-averaging over a wave period. The mean velocity $U_i (i = 1, 2)$ and W can be defined as

$$U_i(\boldsymbol{x}, t) = \frac{1}{h + \langle \zeta \rangle} \left\langle \int_{-h}^{\zeta} u_i(\boldsymbol{x}, z, t) \mathrm{d}z \right\rangle,$$

$$W(\boldsymbol{x}, t) = \frac{1}{h + \langle \zeta \rangle} \left\langle \int_{-h}^{\zeta} w(\boldsymbol{x}, z, t) \mathrm{d}z \right\rangle. \quad (3)$$

Upon averaging the mass-conservation equation and the Navier-Stokes equations of motion for a uniform incompressible viscous fluid over a wave period after integration with respect to z, we obtain the following averaged equations.

The averaged mass equations $(i = 1, 2)$:

$$\frac{\partial \langle \zeta \rangle}{\partial t} + \frac{\partial}{\partial x_i} [(h + \langle \zeta \rangle) U_i] = 0. \quad (4)$$

The averaged horizontal and vertical momentum equations $(i, j = 1, 2)$:

$$\rho(h + \langle \zeta \rangle) \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + g \frac{\partial \langle \zeta \rangle}{\partial x_i} \right) + \frac{\partial S_{ij}}{\partial x_j}$$
$$= [\langle p^b \rangle - \rho g (h + \langle \zeta \rangle)] \frac{\partial h}{\partial x_i}$$
$$+ \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} \sigma'_{ji} \mathrm{d}z \right\rangle + \langle \tau_i^s \mid \nabla R \mid \rangle$$
$$- \langle \tau_i^b \mid \nabla B \mid \rangle, \quad (5a)$$

$$\rho(h + \langle \zeta \rangle) \left(\frac{\partial W}{\partial t} + U_j \frac{\partial W}{\partial x_j} \right)$$
$$+ \left\langle \frac{\partial}{\partial x_j} \int_{-h}^{\zeta} (\rho \tilde{u}_j \tilde{w} - \sigma'_{j3}) \mathrm{d}z \right\rangle$$
$$= \langle p^b \rangle - \rho g (h + \langle \zeta \rangle)$$
$$+ \langle \tau_3^s \mid \nabla R \mid \rangle - \langle \tau_3^b \mid \nabla B \mid \rangle. \quad (5b)$$

The averaged total energy equations $(i, j = 1, 2;\ k, m = 1, 2, 3)$:

$$\frac{\partial E}{\partial t} + \frac{\partial F_i}{\partial x_i} = - Q, \quad (6)$$

where a number of averaged quantities read:

(i) the total energy density:

$$E = \tilde{E} + (h + \langle \zeta \rangle) \left(\frac{1}{2} \rho V^2 + \rho g \langle \zeta \rangle \right)$$
$$- \frac{1}{2} \rho g (h + \langle \zeta \rangle)^2,$$

(ii) the fluctuating energy density:

$$\tilde{E} = \left\langle \int_{-h}^{\zeta} \frac{1}{2} \rho \tilde{v}^2 \mathrm{d}z \right\rangle + \frac{1}{2} \rho g \langle \tilde{\zeta}^2 \rangle,$$

(iii) the total energy flux:

$$F_i = \tilde{F}_i + U_i \tilde{E} + U_j S_{ij}$$
$$+ U_i (h + \langle \zeta \rangle) \left(\frac{1}{2} \rho V^2 + \rho g \langle \zeta \rangle \right)$$
$$- (h + \langle \zeta \rangle) U_j \langle \sigma'_{ji} \rangle + W \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} \mathrm{d}z \right\rangle,$$

(iv) the fluctuating energy flux:

$$\tilde{F}_i = \left\langle \int_{-h}^{\zeta} \mathrm{d}z \left[\tilde{u}_i \left(\frac{1}{2} \rho \tilde{v}^2 + p \right) - \tilde{v}_k \tilde{\sigma}'_{ki} \right] \right\rangle$$
$$+ \left\langle \int_{-h}^{\zeta} \tilde{u}_i \rho g (z - \langle \zeta \rangle) \mathrm{d}z \right\rangle,$$

(v) the radiation stress:

$$S_{ij} = \left\langle \int_{-h}^{\zeta} (\rho \tilde{u}_i \tilde{u}_j + p \delta_{ij}) \mathrm{d}z \right\rangle - \frac{1}{2} \rho g (h + \langle \zeta \rangle)^2 \delta_{ij},$$

(vi) the dissipative terms:

$$Q = - v_k^s \tau_k^s \mid \nabla R \mid + v_k^b \tau_k^b \mid \nabla B \mid + \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} \mathrm{d}z.$$

In addition, the viscous stress tensor

$$\sigma'_{km} = \langle \sigma'_{km} \rangle + \tilde{\sigma}'_{km},$$

$$\boldsymbol{v} = \boldsymbol{V} + \tilde{\boldsymbol{v}} \equiv (U_1, U_2, W) + (\tilde{u}_1, \tilde{u}_2, \tilde{w}),$$

τ_k^s and τ_k^b denote respectively the stresses acting on the free surface by the wind and the bottom shear stress exerted by the fluid, $R(\boldsymbol{x}, z, t)$ and $B(\boldsymbol{x}, z)$ represent the free surface and the bottom surface respectively, the bottom pressure $p^b \equiv p(\boldsymbol{x}, -h, t)$, $\nabla = \left(\frac{\partial}{\partial \boldsymbol{x}}, \frac{\partial}{\partial z} \right)$.

If we subtract from the overall energy equation (6), $\rho g \langle \zeta \rangle$ times (4) together with U_i times (5a) and W times (5b), we obtain an energy equation for the fluctuating motion

$$\frac{\partial \tilde{E}}{\partial t} + \frac{\partial}{\partial x_i} \left\{ \left\langle \int_{-h}^{\zeta} \mathrm{d}z \left[\tilde{u}_i \left(\frac{1}{2} \rho \tilde{v}^2 + p \right. \right. \right. \right.$$
$$\left. \left. \left. \left. + \rho g (z - \langle \zeta \rangle) \right) \right] \right\rangle + U_i \tilde{E} \right\}$$
$$+ \frac{\partial W}{\partial x_i} \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} \mathrm{d}z \right\rangle + \left(W + U_i \frac{\partial h}{\partial x_i} \right)$$
$$\cdot \left[\rho (h + \langle \zeta \rangle) \left(\frac{\partial W}{\partial t} + U_j \frac{\partial W}{\partial x_j} \right) \right.$$
$$\left. + \frac{\partial}{\partial x_i} \left\langle \int_{-h}^{\zeta} \rho \tilde{u}_i \tilde{w} \mathrm{d}z \right\rangle \right] + S_{ij} \frac{\partial U_j}{\partial x_i} + D = 0, \quad (7)$$

where D denotes the total dissipative effects and takes the form

$$D = \langle \tilde{v}_k^b \tau_k^b \rangle \mid \nabla B \mid - \langle \tilde{v}_k^s \tau_k^s \rangle \mid \nabla R \mid$$
$$+ \left\langle \int_{-h}^{\zeta} \sigma'_{km} \frac{\partial v_k}{\partial x_m} \mathrm{d}z \right\rangle + U_i \frac{\partial}{\partial x_j} [(h + \langle \zeta \rangle) \langle \sigma'_{ji} \rangle]$$
$$- U_i \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j} \left\langle \int_{-h}^{\zeta} \sigma'_{j3} \mathrm{d}z \right\rangle$$
$$+ \left(U_i \frac{\partial h}{\partial x_i} + W \right) [\langle \tau_3^b \mid \nabla B \mid \rangle - \langle \tau_3^s \mid \nabla R \mid \rangle]$$
$$- \frac{\partial}{\partial x_i} \left\{ \left\langle \int_{-h}^{\zeta} \tilde{v}_k \tilde{\sigma}'_{ki} \mathrm{d}z \right\rangle + (h + \langle \zeta \rangle) U_j \langle \sigma'_{ji} \rangle \right\}. \quad (8)$$

2 Generalized wave action conservation equation

In classical mechanics a ratio of energy to frequency is called action and found to be invariant when the setting is theory of slow modulation for vibrating systems. In the case of wave, if the medium (such as ambient currents and bottoms) varies slowly with x and t, then we arrive at the classical wave action conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{\tilde{E}'}{\omega_r}\right)+\frac{\partial}{\partial x_i}\left[\left(U_i+C_{gi}\right)\frac{\tilde{E}'}{\omega_r}\right]=0, \qquad (9)$$

in which wave action $A'=\frac{\tilde{E}'}{\omega_r}$ is the energy density $\tilde{E}'$ divided by intrinsic angular frequency ω_r, C_{gi} is the group velocity relative to U_i.

Within the framework of geometric-optic approximation, in Eq. (7) adding and subtracting the term: $\frac{1}{\omega_r}\frac{\partial \omega_r}{\partial x_i}\tilde{E}(U_i+C_{gi})$, and dividing the result by ω_r, yields

$$\begin{aligned}&\left\{\frac{\partial A}{\partial t}+\frac{\partial}{\partial x_i}\left[A\left(U_i+C_{gi}\right)\right]\right\}\\&+\frac{1}{\omega_r}\left\{\frac{\partial W}{\partial x_i}\left\langle\int_{-h}^{\zeta}\rho\tilde{u}_i\tilde{w}\,\mathrm{d}z\right\rangle\right.\\&+\left(W+U_i\frac{\partial h}{\partial x_i}\right)\left[\rho\left(h+\langle\zeta\rangle\right)\left(\frac{\partial W}{\partial t}+U_j\frac{\partial W}{\partial x_j}\right)\right.\\&\left.\left.+\frac{\partial}{\partial x_i}\left\langle\int_{-h}^{\zeta}\rho\tilde{u}_i\tilde{w}\,\mathrm{d}z\right\rangle\right]\right\}+\frac{D}{\omega_r}\\&+\left\{\frac{A}{\omega_r}\left[\frac{\partial\omega_r}{\partial t}+\left(U_i+C_{gi}\right)\frac{\partial\omega_r}{\partial x_i}\right]\right.\\&\left.+\frac{1}{\omega_r}S_{ij}\frac{\partial U_i}{\partial x_i}\right\}=0, \qquad (10)\end{aligned}$$

where $A=\frac{\tilde{E}}{\omega_r}$. Among the four terms on the left-hand side of (10), the first term constitutes the basic form of the classical wave action conservation equation (9), the second term can be defined as the vertical velocity wave action and the third term the dissipation wave action, finally the fourth term vanishes identically[4]. Thus Eq. (10) reduces to an extensive conservation form . We can show[4] the equivalence of Eq. (9) for many other types of wave motion in fluid dynamics, therefore, Eq. (10) can be regarded as a valuable extension to (9), and named a generalized wave action conservative equation for the dissipative dynamical system in the nearshore region, which will play an important role in dealing with the process of real viscous flow.

A tentative relation between wave action conservation and dissipative effects was given by Christoffersen and Jonsson[10].

3 Concluding remarks

Starting from the Navier-Stokes equations, we obtain a generalized wave action conservation equation and thus put forward two new concepts for wave action, extending the classical concept of wave action from the ideal averaged flow conservative system into the real averaged flow dissipative system and revealing the richness and variety of wave action. If we want to consider a special dissipative effect (such as breaking effect), it then can be added to the dissipative wave action, and the generalized conservative form of Eq. (10) remains unchanged.

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